Design of Robust Controller for an Uncertain System described by Unstructured Uncertainty Model using Small Gain Theorem

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***Abstract*-- A simple and easy technique is used to design a robust Proportional Integral Derivative (PID)/ Proportional Integral (PI) controller for an uncertain system described by unstructured uncertainty model. The controller design is based on Small Gain theorem and it guarantees closed loop stability for input variations across its full working range. The uncertain plant under different working ranges is described by a family of corresponding transfer functions. In this paper design of a robust controller has been demonstrated for a warm air drying chamber. Closed loop step response and frequency response plots are shown to prove that the proposed controller provides robust operation of the chosen uncertain plant. Comparison of the results with those obtained from other design methods shows that the proposed controller design methodology proves to be a better option.**

Keywords—Additive uncertainty, PID controller, Robust control, Small Gain theorem

I. Introduction

A mathematical model used to describe any physical system is always only an approximate one since many factors such as reduction in system order, time delay, non-linearity, parameter variations due to disturbances etc. are ignored in the process of model simplification. This may result in modeling error and unpredictable plant behavior which necessitates the need for uncertainty modeling. Uncertainty due to dynamic perturbations is a result of deviation of the model from the actual system dynamics and uncertainty due to disturbance signals is because of input and output disturbances.

Unstructured uncertainty uses a single perturbation block to represent dynamic perturbations such as those due to neglected plant dynamics or unmodeled high frequency dynamics. Representation of this type of uncertainty is possible in many ways.

Additive perturbation model is represented by the equation $G_p(s) = G_0(s) + \Delta(s)$ and the equation $G_p(s) = G_0(s)[1+\Delta(s)]$ represents the multiplicative perturbation model. $G_p(s), G_0(s)$ and $\Delta(s)$ denote the actual perturbed plant dynamics, the nominal plant model and the uncertainty model respectively. The block $\Delta(s)$ is normally bounded i.e. $\bar{\sigma}[\Delta(j\omega)] \le \delta(j\omega)\,\forall\,\omega \in [0,\infty]$ where $\bar{\sigma}$ represents the largest singular value of the matrix and $\delta(j\omega)$ is a known scalar transfer function[1]. Generally, conventional controller design methodologies require an accurate plant model to achieve the desired performance. Since practical systems which are sensitive to parameter variations result in model uncertainty, designing a controller that stabilizes and meets performance requirements for such systems poses a challenge. Proportional Integral Derivative (PID) controllers have gained popularity in industries since they are easily implementable. However, since they make use of heuristic tuning methods such as Zeigler-Nichols, they lack generality [2].

Designing a Proportional Integral (PI)/PID controller for uncertain systems in a simple way using Small Gain theorem has been proposed in [3]. [4] presents a simple procedure for PID controller design for nonlinear uncertain systems of second order. For uncertain plants with parameters bounded, Kharitonov theorem based design of a robust PID controller has been demonstrated in [5]. In [6], design of robust PI controller using Kharitonov theorem has been carried out for an interval plant with bounded but unknown time delay. A robustly stabilizing PID controller for an oblique wing aircraft with parameters having interval uncertainty has been designed using Kharitonov theorem in [7]. In [8], a robust PID controller design methodology based on Kharitonov theorem is proposed for an uncertain first order plus time delay system with gain and phase margin constraints. A robust PID controller using second order sliding mode technique is designed for unmatched uncertain systems under the influence of disturbance signals [9]. Control problem has been solved as an optimization problem using H_∞ framework in [10], [11], [12], [13].

Robust controllers have been designed using the D curves method for a linear interval model and using the D partition method for an affine model of an uncertain warm air drying chamber in [14]. These approaches are tedious and time consuming since they are graphical methods based on Kharitonov theorem. A simple frequency domain based approach to design a robust PID/PI controller for an industrial warm air drying chamber described by a family of transfer functions has been proposed in this article.

II. Design Methodology

Small Gain theorem classifies a closed loop system as robustly stable if the magnitude of the open loop gain is less than 1. In other words, if $|L(s)| < 1\,\forall\,\omega \in [0,\infty]$ then the system will be closed loop stable. A family of stable transfer functions $G_k(s)$ is used to describe an uncertain plant to be controlled. Then the plant can be represented by an additive unstructured uncertainty model as shown in Fig. 1. The additive unstructured uncertainty model of the uncertain plant is given by,

$$G(s) = G_0(s) + W_a(s).\Delta(s) \qquad (1)$$

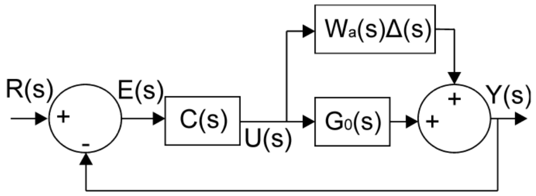

Fig. 1. Closed loop system with plant described by additive uncertainty model.

where $G_0(s)$ is the nominal transfer function, $W_a(s)$ is the additive weighting transfer function and $\Delta(s)$ is a set of transfer functions such that their peak magnitudes are less than or equal to 1 for all frequencies. If $\Delta(s) \le 1$ then, $|W_a(s).\Delta(s)| \le |W_a(s)|$. This implies that,

$$|W_a(s)| \ge Max|G_k(s) - G_0(s)| \,\forall\, \omega \in [0, \infty] \tag{2}$$

where k = 1, 2, 3... n and $G_k(s)$ is the family of stable transfer functions representing the uncertain system. Based on (2), a suitable choice of the weighting transfer function is made.

The closed loop system shown in Fig. 1 has its characteristic equation as

$$1 + C(s)G(s) = 0 \tag{3}$$

Using (1) in (3), we have,

$$1 + C(s)[G_0(s) + W_a(s).\Delta(s)] = 0 \tag{4}$$

Rearranging,

$$[1 + C(s)G_0(s)]\left[1 + \frac{C(s)G_0(s)}{1 + C(s)G_0(s)} . \frac{W_a(s)\Delta(s)}{G_0(s)}\right] = 0 \tag{5}$$

Let the nominal closed loop transfer function given by,

$$M_0(s) = \frac{C(s)G_0(s)}{1 + C(s)G_0(s)} \tag{6}$$

be assumed stable.

Using (6) in (5),

$$[1 + C(s)G_0(s)]\left[1 + M_0(s). \frac{W_a(s)\Delta(s)}{G_0(s)}\right] = 0 \tag{7}$$

Assuming stability of nominal closed loop system, the nominal characteristic equation given by (3) is also stable.

The necessary and sufficient condition for system shown in Fig. 1 to be stable is that (7) should be stable. Hence the equations $[1 + C(s)G_0(s)] = 0$ and

$\left[1 + M_0(s). \frac{W_a(s)\Delta(s)}{G_0(s)}\right] = 0$ should also be stable.

$\left[1 + M_0(s). \frac{W_a(s)\Delta(s)}{G_0(s)}\right] = 0$ will be stable if it satisfies the condition of Small Gain theorem i.e., magnitude of gain of open loop should be less than one. Hence the following condition must be satisfied.

$$\left|M_0(s). \frac{W_a(s)\Delta(s)}{G_0(s)}\right| < 1 \,\forall\, \omega \in [0, \infty] \tag{8}$$

Considering the worst case value of $\Delta(s)$, that is,

$|\Delta(s)| = 1 \,\forall\, \omega \in [0, \infty]$, (8) becomes,

$$|M_0(s)| < \left|\frac{G_0(s)}{W_a(s)}\right| \tag{9}$$

Based on (9), a suitable choice for the nominal closed loop transfer function $M_0(s)$ can be made as follows. Rearranging (6) we have,

$$C(s) = \frac{M_0(s)}{G_0(s)[1 - M_0(s)]} \tag{10}$$

Expressing the transfer functions as ratios of pertinent polynomials that is, relevant or appropriate numerator and denominator polynomials, each of the controller transfer functions in (10), is given by,

$$\frac{C_n(s)}{C_d(s)} = \frac{\dfrac{M_{0n}(s)}{M_{0d}(s)}}{\dfrac{G_{0n}(s)}{G_{0d}(s)}\left[1 - \dfrac{M_{0n}(s)}{M_{0d}(s)}\right]} \tag{11}$$

Subscripts n and d in (11) represent the respective numerator and denominator polynomials of the transfer functions.

$$\frac{C_n(s)}{C_d(s)} = \frac{G_{0d}(s)M_{0n}(s)}{G_{0n}(s)[M_{0d}(s) - M_{0n}(s)]} \tag{12}$$

Similarly the transfer function in (6) is also expressed in terms of ratios of pertinent polynomials, resulting in (13).

$$\frac{M_{0n}(s)}{M_{0d}(s)} = \frac{\dfrac{C_n(s)}{C_d(s)} . \dfrac{G_{0n}(s)}{G_{0d}(s)}}{1 + \dfrac{C_n(s)}{C_d(s)} . \dfrac{G_{0n}(s)}{G_{0d}(s)}} \tag{13}$$

Rearranging (13),

$$\frac{M_{0n}(s)}{M_{0d}(s)} = \frac{G_{0n}(s)}{C_d(s)\left[\dfrac{G_{0d}(s)}{C_n(s)}\right] + G_{0n}(s)} \tag{14}$$

Let

$$\frac{G_{0d}(s)}{C_n(s)} = P_0(s) \tag{15}$$

Using (15) in (14) we have,

$$\frac{M_{0n}(s)}{M_{0d}(s)} = \frac{G_{0n}(s)}{C_d(s)[P_0(s)] + G_{0n}(s)} \tag{16}$$

Since the transfer functions in (13), (14) and (16) are expressed in terms of ratios of pertinent numerator and denominator polynomials, the numerators and denominators on both sides of (16) can be equated and we have,

$$M_{0n}(s) = G_{0n}(s) \tag{17}$$

$$M_{0d}(s) = C_d(s)P_0(s) + G_{0n}(s) \tag{18}$$

Using (17) in (12), controller transfer function can be written as,

$$\frac{C_n(s)}{C_d(s)} = \frac{G_{0d}(s)}{[M_{0d}(s) - G_{0n}(s)]} \tag{19}$$

Conventional PID controller transfer function is given by,

$$\frac{C_n(s)}{C_d(s)} = \frac{1}{K}\left(\frac{X(s)}{s}\right) = \frac{1}{K}\left(\frac{x_d s^2 + x_p s + x_i}{s}\right) \quad (20)$$

The choice of controller numerator and controller denominator polynomials are made as explained in the following. Rearranging (18) we have,

$$M_{0d}(s) - G_{0n}(s) = C_d(s) P_0(s) \quad (21)$$

From (15) we have,

$$G_{0d}(s) = P_0(s) C_n(s) \quad (22)$$

Using (21) and (22) in (19) the controller transfer function changes to,

$$\frac{G_{0d}(s)}{[M_{0d}(s) - G_{0n}(s)]} = \frac{P_0(s) C_n(s)}{C_d(s) P_0(s)} \quad (23)$$

Using PID controller conventional transfer function given by (20) in (23),

$$\frac{G_{0d}(s)}{[M_{0d}(s) - G_{0n}(s)]} = \frac{P_0(s)}{P_0(s)} \frac{1}{K}\left(\frac{X(s)}{s}\right) \quad (24)$$

Using the notion of pertinent numerator and denominator polynomial representation for the transfer function, the numerator and denominator on either side of (24) can be equated. That is, equating the numerator polynomials on either side of (24),

$$G_{0d}(s) = P_0(s) X(s) = P_0(s)\left(x_d s^2 + x_p s + x_i\right) \quad (25)$$

and equating the denominator polynomials on either side of (24),

$$M_{0d}(s) - G_{0n}(s) = P_0(s).K.s \quad (26)$$

In (25), the degree of polynomial $P_0(s)$ is chosen such that it is equal to the degree of polynomial $G_{0d}(s)$ minus the degree of polynomial $X(s)$. Likewise $P_0(s)$ and $X(s)$ can be determined. Using (21) and (26) in (16), an expression for the nominal closed loop transfer function $M_0(s)$ is obtained as a function of K as in (27).

$$M_0(s) = \frac{M_{0n}(s)}{M_{0d}(s)} = \frac{G_{0n}(s)}{P_0(s).K.s + G_{0n}(s)} \quad (27)$$

Now, the controller design has been transformed in to choosing a suitable K such that the condition given by (9) as obtained through (1) to (9) is satisfied.

With the selected value of K, the controller parameters are determined using (28).

$$K_d = \frac{x_d}{K}, \quad K_p = \frac{x_p}{K} \quad \text{and} \quad K_i = \frac{x_i}{K} \quad (28)$$

III. Controller Design for the Uncertain System

The example of warm air drying chamber used in [14] has been considered as an uncertain system to demonstrate the applicability of the proposed controller design method. The transfer functions for the uncertain system that have been obtained by identification of the step responses by Hudzovic method for step change in input power under five different ranges are,

$$G_1(s) = \frac{10.6}{12893s^3 + 8058s^2 + 226.7s + 1} \text{ for } 0 - 20\% \quad (29)$$

$$G_2(s) = \frac{10.5}{2078s^3 + 10346s^2 + 230.9s + 1} \text{ for } 20 - 40\%$$

$$G_3(s) = \frac{11.4}{40184s^3 + 10506s^2 + 255.6s + 1} \text{ for } 40 - 60\%$$

$$G_4(s) = \frac{12.8}{13681s^3 + 11606s^2 + 279.4s + 1} \text{ for } 60 - 80\%$$

$$G_5(s) = \frac{8}{32324s^3 + 10370s^2 + 276.97s + 1} \text{ for } 80 - 100\%$$

The nominal model for the uncertain plant is given by,

$$G_0(s) = \frac{10.66}{20232s^3 + 1.1077\times 10^4 s^2 + 253.9s + 1} = \frac{G_{0n}(s)}{G_{0d}(s)} \quad (30)$$

A suitable choice for the weighting transfer function is made so as to satisfy the condition given by (2) using the function `ltiarray2uss()` in Matlab®.

$$W_a(s) = \frac{9.63\times 10^{-6} s^3 + 4.75\times 10^{-5} s^2 + 9.44\times 10^{-4} s + 1.44\times 10^{-4}}{s^3 + 0.5719 s^2 + 99.2\times 10^{-4} s + 5.106\times 10^{-5}} \quad (31)$$

Fig. 2 shows the frequency response of the chosen weighting transfer function $W_a(s)$ and the five difference transfer functions $|G_1(s) - G_0(s)|$, $|G_2(s) - G_0(s)|$, $|G_3(s) - G_0(s)|$, $|G_4(s) - G_0(s)|$ and $|G_5(s) - G_0(s)|$.

From (30), it can be seen that $G_{0d}(s)$ is a third degree polynomial. Then from (25), it is obvious that $P_0(s)$ should be a polynomial of first degree. Hence let $P_0(s) = p_1 s + 1$. Now (25) becomes,

$$20232s^3 + 1.1077\times 10^4 s^2 + 253.9s + 1 = (p_1 s + 1)(x_d s^2 + x_p s + x_i) \quad (32)$$

Equating respective powers of s coefficients on both sides of (32) we obtain,

$$x_d = 9652.19;\ x_p = 251.804;\ x_i = 1;\ p_1 = 2.096 \quad (33)$$

Next, $M_0(s)$ and $|G_0(s)/W_a(s)|$ are plotted for different values of K. It is found that the minimum value of K that satisfies the robust stability condition given by (9) is 19. If K is increased beyond 19, the peak overshoot decreases. Here, K = 60 is chosen which gives a small, maximum value of peak overshoot of 12.3%. For ease of choosing the value of K, a Matlab® program using for-loop which plots frequency response curves for $\left|\frac{G_0(s)}{W_a(s)}\right|$ and a series of curves for nominal closed loop gain for a range of values of K is used.

Fig. 3 shows the frequency response of $|G_0(s)/W_a(s)|$ and $M_0(s)$ for $K = 60$. Using the chosen value of K and (33) in (28), the PID controller parameters are determined.

$$K_p = 4.2; \quad K_d = 160.9; \quad K_i = 0.0167 \quad (34)$$

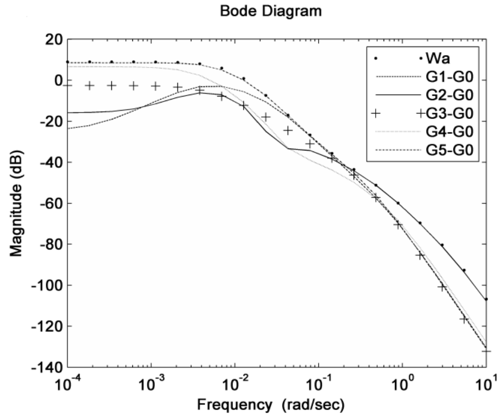

Fig. 2. Frequency response of W_a and $\left|G_k - G_0\right|$.

IV. Simulation Results

A. Comparison with PID controller by D curves method

The step and frequency responses of the closed loop warm air-drying chamber with the proposed PID controller are compared with those of the PID controller designed using the D curves method [14]. The step responses with the proposed PID controller and that designed using D curves method [14] are presented in Fig. 4. From the results obtained it is very clear that the proposed controller resulted in step responses, with less overshoot and fast settling time for all regions of operation respectively, which are quantitatively compared in Table 1.

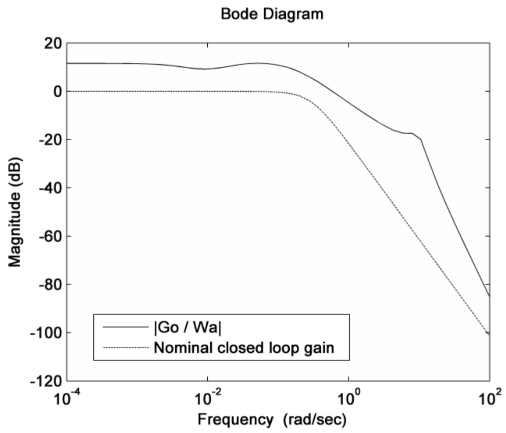

Fig. 3. Frequency response of $\left|G_0 / W_a\right|$ and M_0.

However the rise times lie within 12.8 seconds, which is higher than those for D curves method. If K is chosen such that the maximum rise time is 7.61 seconds then maximum % peak overshoot and settling times are 25.68 and 28.74 seconds respectively which indicates a better performance.

Fig. 5 shows the corresponding frequency responses with the proposed PID controller giving a higher value of minimum phase margin of 55.9 degrees (for G_3) compared to that with D curves method giving a minimum phase margin of 34 degrees.

TABLE I. Comparison between proposed PID controller and PID controller by D Curves method [14]

Time response specifications	*Proposed PID controller*	*PID controller by D curves method*
Maximum rise time (sec)	12.80	7.61
Maximum % peak overshoot	12.31	47.33
Maximum settling time (sec)	27.09	64.01

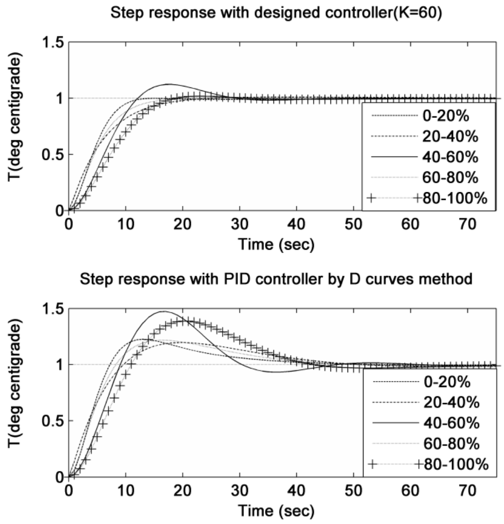

Fig. 4. Step response with proposed PID controller and with PID controller by D curves method [14].

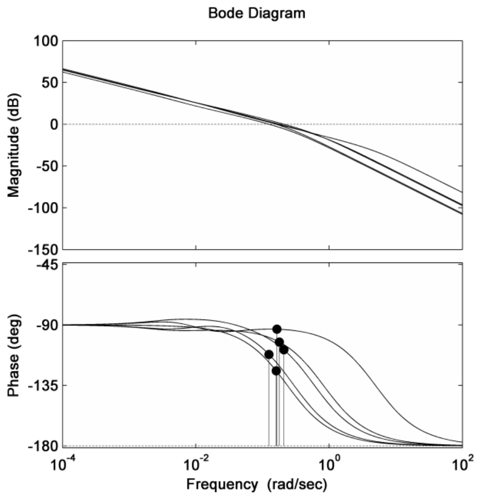

Fig. 5. Frequency response with the proposed PID controller (K=60).

B. Comparison with PI controller by D curves method

[14] shows the design of a PI controller by D curves method with a demanded phase margin of 70 degrees. Hence a PI controller is designed using the proposed methodology based on Small Gain theorem to achieve this phase margin. K=1900 satisfies the above design criteria. The parameters of PI controller are determined as,

$$K_p = 0.1077; \quad K_i = 0.000526 \tag{35}$$

Fig. 6 shows the step responses obtained with the proposed PI controller and those with the PI controller by D curves method [14]. Table 2 shows that the proposed controller gives a better time response with improved peak over shoot and settling time compared with marginally higher rise time.

TABLE II. Comparison between proposed PI controller and PI controller by D Curves method [14]

Time response specifications	*Proposed PI controller*	*PI controller by D curves method*
Maximum rise time (sec)	420.98	1523.20
Maximum % peak overshoot	2.30	0.00
Maximum settling time (sec)	680.80	3535.30

C. Comparison with PID controller by D partition approach

Simulation of the PID controller designed by D partition approach as in [14] gives a minimum phase margin of 45.8 degrees. Now the parameter K is chosen such that this minimum phase margin requirement is achieved in addition to a reduced peak overshoot. K=120 satisfies this criteria. The PID controller parameters determined are given by,

$$K_p = 2.0984; \quad K_d = 80.4349; \quad K_i = 0.0083 \tag{36}$$

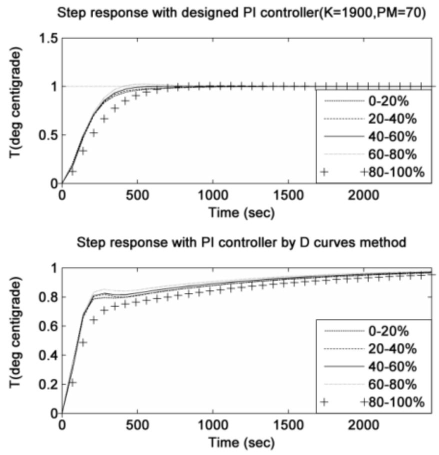

Fig. 6. Step response with proposed PI controller and with PI controller by D curves method.

Fig. 7 shows improved step responses with the proposed PID controller as compared to that with the PID controller by D partition approach.

Table 3 shows that the proposed controller gives a better time response that is, it is fast, has a small % peak overshoot and settles down quickly as compared to the PID controller by D partition approach.

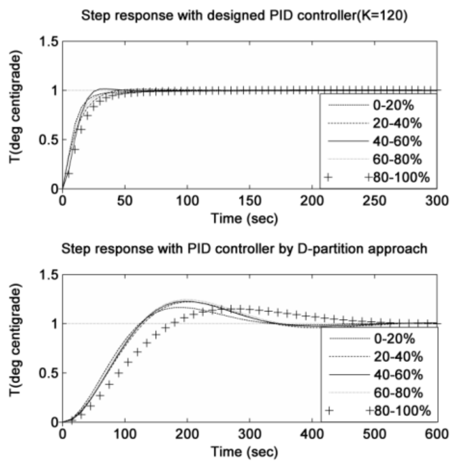

Fig. 7. Step response with proposed PID controller and PID controller by D partition approach.

TABLE III. COMPARISON BETWEEN PROPOSED PID CONTROLLER AND PID CONTROLLER BY D PARTITION APPROACH [14]

Time response specifications	*Proposed PID controller*	*PID controller by D partition approach*
Maximum rise time (sec)	28.44	118.14
Maximum % peak overshoot	1.66	24.55
Maximum settling time (sec)	59.86	490.38

V. CONCLUSION

The paper presented a simple and easy to design methodology of robust PID/PI controller using small gain theorem for systems having unstructured uncertainty. The proposed controller design method has been applied to a warm air-drying chamber presented in [14] successfully. The results of plant operation for a temperature change between 0 and 100 % of the maximum working range show that the designed controller results in robust performance for all the five operating regions. The proposed controller's performance is better with respect to time and frequency responses compared to the D curves and D partition methods. However the proposed technique can be applied only to uncertain systems whose nominal model transfer function is stable. The Small Gain theorem condition ensures that the effect of perturbations occurring in the uncertain system is eliminated and results in stable closed loop operation. Further, this controller design technique is applicable only to uncertain systems with order less than three. Also, the proposed method does not give optimal parameters. However it is shown that it results in both better overshoot and settling time. As a future scope, the proposed method may be modified to obtain optimal parameters.

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